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State Transition Matrices for Terminal Rendezvous Studies: Brief Survey and New Example

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A brief survey and classification of much of the published material on linearized rendezvous is presented. This is followed by a new form of solution of the terminal rendezvous problem that is valid in a general central force field. This solution and the solution of the related adjoint system are used to construct a general state transition matrix. Because of the generality of the assumptions, this state transition matrix is very concise and flexible. Finally, the work is applied to the problem of terminal rendezvous near any Keplerian orbit in a Newtonian gravitational field using the Tschauner–Hempel equations. Because this solution is presented in terms of the true anomaly, considerable care is taken to avoid the types of singularities that are typical in this kind of problem. The result is a state-transition matrix for linearized rendezvous studies that is thought to be simpler and more convenient than other versions found in the literature.

I. Introduction

LINEARIZED equations of motion are useful in describing the terminal rendezvous phase of a mission or in satellite station keeping. These areas of astrodynamics are rich in the variety of linearizations available to investigators and in the resulting mathematical analysis and computations that follow. This paper presents a brief survey of the types of linear models found in rendezvous studies, followed by a new general model that incorporates much previous work as special cases. The work combines some ideas in 19th century celestial mechanics with some recent discoveries. Because this new model assumes a general central force gravitational field, much of the complexities found in specific cases are avoided in designing a relatively simple state transition matrix that is applicable to a variety of problems.

The work is then applied to the important special case of linearized rendezvous in a gravitational field defined by an inverse square law using the Tschauner–Hempel equations. In fact, it was this problem that motivated the study. The search for a solution devoid of singularities, valid for any Keplerian orbit, that avoids universal functions can lead to a cluttered set of equations. The new general model avoids much of this clutter and presents the results in a relatively concise form.

II. Survey of Linearized Rendezvous

A. Early Work

Some of the mathematical principles of linearized rendezvous, often called terminal rendezvous or proximity equations, go back far into the literature of celestial mechanics. In the study of perturbations caused by disturbing forces, variational equations are considered that describe the deviations of a mass from a nominal orbit. A good example can be found in Hill's work.[1] Also, in describing the equations of the moon relative to the Earth, Hill's equations[2] contain a term μ/r^3 that, if removed, results in linear equations now commonly called the Clohessy–Wiltshire equations.

B. Classification of Linearized Rendezvous Studies

The literature contains many linearized models used for terminal rendezvous studies. These can be classified in various ways. 1) The linearized equations can be presented in an inertial frame, or, as is more common, in a rotating frame. The axes of the rotating frame may or may not be orthogonal. 2) The process may involve linearizing the gravitational force function resulting in a gravity gradient, with the equations described in either rectangular or polar coordinates, or the linearization can be performed with respect to various types of parameters associated with an orbit such as Lagrange's orbital elements. 3) Finally, the studies may be classified according to the type of nominal orbit under consideration. In most studies the nominal orbit is circular or near circular, although studies involving high-eccentricity elliptic orbits can be found. Use of other Keplerian orbits is rare. The most extensive use of linearized studies about points in non-Keplerian orbits can be found in the linearized equations of motion with respect to a libration point in the restricted three-body problem. Figure 1 contains an outline of ways of classifying the studies represented in this brief survey. The survey is not intended to be exhaustive or uniform.

C. Linearization of Gravitation in Inertial Frame

The most direct approach to linearized equations of motion is to linearize the gravitational force function in an inertial coordinate frame. This is found in Battin's[3] early book and applied to recursive navigation theory. In his later book he presents this work in a more general framework.[4] This model was used by Lion[5] and Lion and Handelsman[6] in their development of the properties of the primer vector.

D. Circular Orbits

Studies of rendezvous or transfer near a circular orbit appear to be the most numerous in the literature. Although Edelbaum's paper[7] in 1967 uses linearization with respect to the orbital elements and Jones' rendezvous study[8] uses this same model, the vast majority of these studies utilize the Clohessy–Wiltshire equations. These equations and various analyses of their solution by Wheelon[9] and Clohessy and Wiltshire[10] appeared in the summer of 1959. They could be found in journal articles by Clohessy and Wiltshire,[11] Geyling,[12] Spradlin,[13] and Eggleston[14] in 1960. London,[15] in 1963, and Anthony and Sasaki,[16] in 1965, extended the work to second order. The Clohessy–Wiltshire equations were used by Tschauner and Hempel[17] to study bounded-thrust maneuvers, by Gobetz[18] to study power-limited maneuvers, and by Prussing,[19,20] Jezewski and Donaldson,[21] and Jezewski[22] for impulsive maneuvers. Relative motion studies by Berreen and Crisp,[23] Carter,[24] and Neff and Fowler[25] emphasized the geometric shape of solutions to the equations. Later these equations were used extensively by various researchers[26–38] as a simple model to explore and investigate ideas or to use as a specific example illustrating a more general result, and they continue to be useful in this respect. Derivation of the Clohessy–Wiltshire equations from more general linearized equations, along with applications to terminal rendezvous maneuvers, can now be found in books on orbital mechanics.[39,40]

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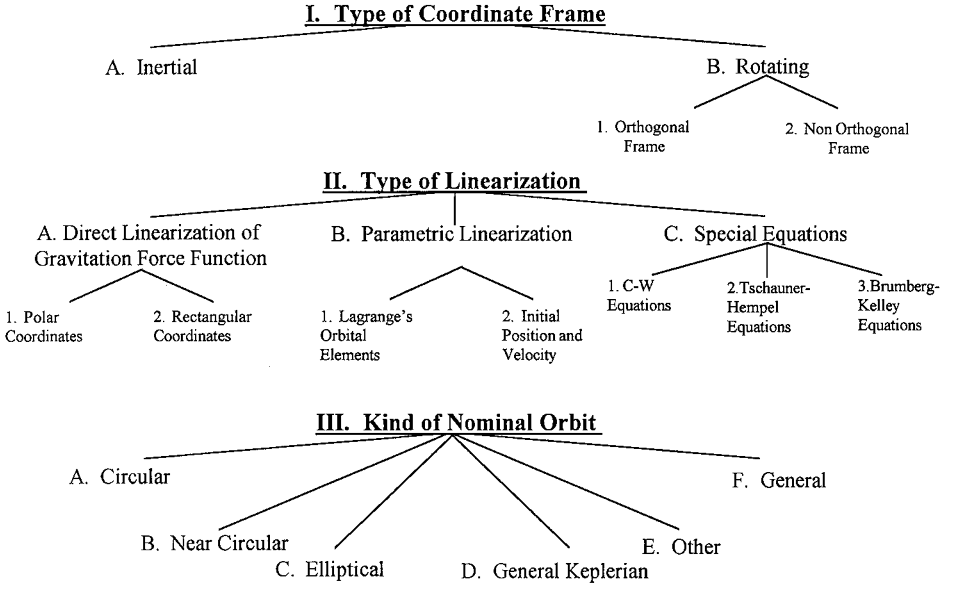

Fig. 1 Ways of classifying linearized rendezvous studies found in the literature.

E. Elliptic and General Keplerian Orbits

Linearized models have been used extensively for noncircular Keplerian orbits although nearly all of this work applies to elliptical orbits.

1. Polar Coordinates

In 1963, Stern[41] linearized the gravitational force function using polar coordinates. These equations were later used by Jones[42] but apparently are not found much in the published literature.

2. Parametric Linearization

Work on parametric linearization is found extensively. Although most of this uses some form of orbital elements, there are also many examples[43–49] in which initial position and velocity vectors are used as parameters. An early example of this approach is found in the work of Pines[43] in 1961. In 1965, Goodyear[44] modified his approach through the use of universal functions. This theme was continued by Shepperd[45] in 1985. Another variation can be seen in the work of Glandorf,[46] who actually inverted a fundamental matrix solution of the adjoint system to find a closed-form state transition matrix for general Keplerian orbits. An improvement by Markley[47] in 1986 avoids the direct inversion of a matrix in the determination of a state transition matrix. Recently, Der[48] and Der and Danchick[49] combined these ideas finding state transition matrices using universal functions without inverting a matrix. In 1965, Gobetz[50] linearized with respect to the orbital elements in one of the earliest studies of power-limited rendezvous near an elliptical orbit. Edelbaum's report[51] in 1969 uses this model. Lancaster,[52] Berreen and Sved,[53] Hestenes,[54] and recently Garrison et al.[55] used linearization with respect to orbital elements to examine relative motion of particles in elliptic orbits. References to work by Marec and others can be found in Marec's book.[56]

3. Tschauner–Hempel Equations

A very convenient set of linearized equations that describes the motion of a spacecraft near a satellite in an arbitrary elliptic orbit relative to a rotating orthogonal coordinate frame fixed in the satellite is the Tschauner–Hempel equations. These generalize the Clohessy–Wiltshire equations and are similar to them in their derivation and types of applications. They were first presented by DeVries[57] in 1963 from the viewpoint of two particles in close elliptical orbits. In this work he only found approximate solutions to these differential equations for low-eccentricity elliptical orbits. In 1965 Tschauner and Hempel[58] derived these equations from the viewpoint of rendezvous of a spacecraft with an object in an elliptical orbit. They found complete solutions for elliptical orbits in terms of the eccentric anomaly. This was followed by additional papers by Tschauner.[59,60] Actually these equations were known by Lawden.[61] They are Eqs. (5.30–5.32) in his book,[61] but they describe the primer vector instead of the spacecraft. It happens that the same differential equations describe the primer vector and the unpowered linearized equations of motion of the spacecraft. This can be seen from Eq. (5.8) of his book, but is found also in his earlier work, e.g., Ref. 62, Eq. (6). In 1966 Shulman and Scott[63] and in 1967 Euler and Shulman[64] presented some numerical studies on the accuracy of solutions of the Tschauner–Hempel equations. Then in 1969, Euler[65] produced the first study of variable exhaust-velocity, power-limited rendezvous utilizing the Tschauner–Hempel equations. In 1981 Weiss[66] found a form of fundamental matrix solution of the Tschauner–Hempel equations for elliptic orbits in terms of the eccentric anomaly that also applies to circular orbits. This work was applied by Wolfsberger et al.[67] to the case of rendezvous with objects in highly eccentric elliptical orbits. Carter and Humi[68] studied these equations for general noncircular Keplerian orbits and showed that the constant exhaust-velocity bounded-thrust fuel-optimal rendezvous problem has no singular solutions. Later, Carter[69] took into account the variation in mass of the spacecraft as fuel is expended. Solutions of the Tschauner–Hempel equations not written using the eccentric anomaly involved the use of an integral

$$I = \int \frac{\mathrm{d}\theta}{\sin^2\theta(1 + e\cos\theta)^2}$$

introduced by Lawden[62] in 1954. This integral consistently appears in Lawden's work[70] and can be found as recently as 1993. The integral is singular wherever the true anomaly θ is a multiple of π. These singularities were removed by Carter,[71] who instead used the integral

$$J = \int \frac{\cos\theta\,\mathrm{d}\theta}{(1 + e\cos\theta)^3}$$

Closed-form evaluation of this integral is best performed using the eccentric anomaly for elliptical orbits or the hyperbolic anomaly for hyperbolic orbits. Van der Ha and Mugellesi[72] presented a derivation of the Tschauner–Hempel equations, but their study of constant radial or circumferential thrust was restricted to the Clohessy–Wiltshire equations. Carter and Brient[73–75] studied both bounded-thrust constant exhaust-velocity and impulsive optimal rendezvous problems for a range of eccentricities including both elliptical and hyperbolic orbits using the Tschauner–Hempel equations. Some unusual effects were found for elliptic orbits with eccentricities near unity.[75] An idealized optimal hyperbolic impulsive rendezvous problem was formulated and solved in closed form by considering the limit of certain forms of solution of the Tschauner–Hempel equations as the eccentricity approaches infinity.[76]

4. Brumberg–Kelley Equations

Another type of linearization similar to that of Tschauner and Hempel was used by Kelley[77] based on an approach by Brumberg.[78] Other references are cited by Kelley.[77] The Brumberg–Kelley linearization differs from the Tschauner–Hempel linearization in two respects. First, time is retained as the independent variable rather than true anomaly. Second, the coordinate axes are not orthogonal, but one axis is directed along the tangent to the orbit and the other along a radial direction from the center of attraction. The resulting equations are not expressed as succinctly as the Tschauner–Hempel equations, but Kelley presents a state transition matrix for elliptical orbits in terms of time and eccentric anomaly. There does not appear to be as much care required in the avoidance of singularities in the solution as in the case of the Tschauner–Hempel equations.

F. General Linear Equations

All of the aforementioned linear models can be thought of as special cases of a general linear system of equations. Various types of fuel-optimal rendezvous problems have been studied using a general linear model. Neustadt[79,80] was one of the first to present results on general linear models. He showed rigorously that the impulsive problem is the limit of the bounded-thrust problem as the bound becomes arbitrarily large. He was also the first to show that the solution of the optimal impulsive problem in n-dimensional state space requires at most n impulses. Simpler and different ways to show this result have been found by Stern and Potter,[81] Carter and Brient,[75] and Prussing.[82] Optimal power-limited rendezvous of general linear systems with bounds on thrust magnitude has been studied by Carter and Pardis.[36–38]

Humi's paper[83] in 1993 presents linear equations of motion of a spacecraft near a satellite in a general central force field and establishes much of the impetus for the present work. The generality of these equations and a method of finding the form of the general solution that was employed by Hill[1] in 1874 lead to a simple and revealing form of the state transition matrix that is especially useful for the Tschauner–Hempel linearization.

III. Rendezvous Equations in a General Central Force Field

Let the vector $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)^T$ represent the position of a spacecraft relative to a satellite in a rotating coordinate frame fixed in the satellite where the superscript T refers to the transpose of a vector or matrix, and let θ represent the true anomaly of the satellite. In any central force field the law of conservation of angular momentum holds:

$$\omega = \dot{\theta} = L/R^2 \tag{1}$$

where the dot represents differentiation with respect to time t, R is the distance of the satellite from the center of the force field, and L is the constant angular momentum of the satellite. A succinct set of linearized equations can be obtained from the transformation to the new vector

$$\boldsymbol{y} = \omega^{\frac{1}{2}}\boldsymbol{x} \tag{2}$$

These equations, obtained by Humi,[83] are as follows:

$$y_1'' = 2y_2' + \omega^{\frac{3}{2}}T_1\big/m \tag{3a}$$

$$y_2'' - G(\omega)y_2 + 2y_1' = \omega^{\frac{3}{2}}T_2\big/m \tag{3b}$$

$$y_3'' + y_3 = \omega^{\frac{3}{2}}T_3\big/m \tag{3c}$$

where the prime indicates differentiation with respect to θ, m is the mass of the spacecraft, $\boldsymbol{T} = (T_1, T_2, T_3)^T$ is the thrust vector of the spacecraft, and

$$G(\omega) = -\frac{\mathrm{d}\psi(R)}{\mathrm{d}R}\frac{R}{\omega^2} \tag{4}$$

where ψ is the central force law. [For a Newtonian force field $\psi(R) = \mu/R^3$ where μ is the universal gravitational constant times the mass of the center of attraction.]

The independent variable t has been replaced by θ in Eqs. (3). For this reason R is regarded as a function of θ. {In the well-known case of a Newtonian force field, $R(\theta) = L^2/[\mu(1 + e\cos\theta)]$.} In general, Eq. (1) becomes

$$\omega(\theta) = L/R(\theta)^2 \tag{5}$$

so that the dependence on θ is also reflected in Eq. (4). Specifically Eqs. (3) become

$$y_1'' - 2y_2' = u_1(\theta) \tag{6a}$$

$$y_2'' - G[\omega(\theta)]y_2 + 2y_1' = u_2(\theta) \tag{6b}$$

$$y_3'' + y_3 = u_3(\theta) \tag{6c}$$

where

$$\boldsymbol{u}(\theta) = [u_1(\theta), u_2(\theta), u_3(\theta)]^T = \omega(\theta)^{\frac{3}{2}}\boldsymbol{T}(\theta)\big/m(\theta) \tag{7}$$

By letting $v(\theta) = [v_1(\theta), v_2(\theta), v_3(\theta)]^T = \boldsymbol{y}'(\theta)$, Eqs. (6) can be put in the following state variable form:

$$\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \\ y_3 \\ v_3 \end{pmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & G[\omega(\theta)] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \\ y_3 \\ v_3 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u_1(\theta) \\ u_2(\theta) \\ u_3(\theta) \end{pmatrix} \tag{8}$$

This is of the general form

$$\boldsymbol{z}' = A(\theta)\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}(\theta) \tag{9}$$

having a well-known solution of the form

$$\boldsymbol{z}(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1}\boldsymbol{z}(\theta_0) + \Phi(\theta)\int_{\theta_0}^{\theta}\Phi(\tau)^{-1}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \tag{10}$$

where $\Phi(\theta)$ is a fundamental matrix solution associated with $A(\theta)$, $\Phi(\theta)^{-1}$ is its inverse, and $\Phi(\theta)\Phi(\theta_0)^{-1}$ is the state transition matrix associated with $A(\theta)$. On an unpowered interval in which $\boldsymbol{u}(\theta) = 0$, we have

$$\boldsymbol{z}(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1}\boldsymbol{z}(\theta_0) \tag{11}$$

The goal is to present the state transition matrix in a convenient form. From $\boldsymbol{z}(\theta)$ one easily obtains $\boldsymbol{y}(\theta)$ and $\boldsymbol{y}'(\theta)$. The actual position of the spacecraft relative to the satellite is found from Eq. (2):

$$\boldsymbol{x}(\theta) = [\omega(\theta)]^{-\frac{1}{2}}\boldsymbol{y}(\theta) \tag{12}$$

To begin this task, the notation $S(\varphi)$ will be used for any antiderivative of a function φ of θ. In applications one may prefer the form

$$S[\varphi(\theta)] = \int_{\theta_0}^{\theta}\varphi(\tau)\,\mathrm{d}\tau \tag{13}$$

although all of the results are valid if any constant is added to this expression.

In terms of this notation Eq. (6a) can be written as

$$y_1' - 2y_2 = S(u_1) + c_3 \tag{14}$$

where c_3 is an arbitrary constant of integration. Substituting y_1' from Eq. (14) into Eq. (6b) results in the second-order differential equation

$$y_2'' + \{4 - \boldsymbol{G}[\omega(\theta)]\}y_2 = -2\boldsymbol{S}[u_1(\theta)] + u_2(\theta) - 2c_3 \tag{15}$$

Analysis and solution of the system (6) reduces basically to that of Eq. (15) because Eq. (14) is of first order and Eq. (6c) represents a forced harmonic oscillator. During unpowered flight, this out-of-plane motion is decoupled from the system.

IV. General Solution of the Linear Second-Order Differential Equation Without a Term Involving the First Derivative

This section considers the second-order differential equation

$$y'' + F(\theta)y = f(\theta) \tag{16}$$

where F is continuous and f is piecewise continuous on a real closed interval $\theta_0 \leq \theta \leq \theta_f$. The results obtained can also be found under weaker assumptions if one is inclined to generalize. The work presented is an elaboration of some of the ideas presented by Hill[1] in 1874.

If φ_1 and φ_2 are any two linearly independent solutions of the homogeneous differential equation

$$y'' + F(\theta)y = 0 \tag{17}$$

then the Wronskian of φ_1 and φ_2, defined as $\varphi_1\varphi_2' - \varphi_2\varphi_1'$, is clearly a constant on $\theta_0 \leq \theta \leq \theta_f$ because $(\varphi_1\varphi_2' - \varphi_2\varphi_1')' = \varphi_1\varphi_2'' - \varphi_2\varphi_1'' = -F\varphi_1\varphi_2 + F\varphi_1\varphi_2 = 0$. Because φ_1 and φ_2 can be multiplied by arbitrary nonzero constants, it is possible to normalize these constants so that

$$\varphi_1(\theta)\varphi_2'(\theta) - \varphi_2(\theta)\varphi_1'(\theta) = 1, \qquad \theta_0 \leq \theta \leq \theta_f \tag{18}$$

Henceforth, φ_1 and φ_2 represent linearly independent solutions of Eq. (17) satisfying Eq. (18).

Consider now a function, reminiscent of the Wronskian but dependent on f, that is defined as follows:

$$w(f) = S(\varphi_1 f)\varphi_2 - S(\varphi_2 f)\varphi_1 \tag{19}$$

Observe that

$$\frac{\mathrm{d}w(f)}{\mathrm{d}\theta} = S(\varphi_1 f)\varphi_2' - S(\varphi_2 f)\varphi_1' \tag{20}$$

$$\frac{\mathrm{d}^2 w(f)}{\mathrm{d}\theta^2} = S(\varphi_1 f)\varphi_2'' - S(\varphi_2 f)\varphi_1'' + f \tag{21}$$

This leads to the following basic result.

Theorem: Let φ_1 and φ_2 be linearly independent solutions of Eq. (17) satisfying Eq. (18), and let $w(f)$ be defined by Eq. (19); then the complete solution of Eq. (16) is given by $c_1\varphi_1 + c_2\varphi_2 + w(f)$, where c_1 and c_2 are arbitrary constants.

Proof: This follows from substituting the expression into Eq. (16) utilizing Eqs. (18–21).

V. Fundamental Matrix Solution

The preceding theorem will be applied to find a complete solution of Eq. (6). Equation (6c) can be disposed of immediately because its complete solution is

$$y_3 = c_5\cos\theta + c_6\sin\theta + S[\cos\theta u_3(\theta)]\sin\theta - S[\sin\theta u_3(\theta)]\cos\theta \tag{22}$$

The theorem is then applied to Eq. (15) where

$$F(\theta) = 4 - G[\omega(\theta)] \tag{23}$$

$$f(\theta) = -2S[u_1(\theta)] + u_2(\theta) - 2c_3 \tag{24}$$

After some manipulation, the complete solution of Eq. (15) can be written as

$$y_2 = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + w(g) \tag{25}$$

where

$$\varphi_3 = 2[\varphi_1 S(\varphi_2) - \varphi_2 S(\varphi_1)] \tag{26}$$

and

$$g = u_2 - 2S(u_1) \tag{27}$$

Substituting Eq. (25) into Eq. (14) and integrating, the complete solution of Eq. (14) is

$$y_1 = 2c_1S(\varphi_1) + 2c_2S(\varphi_2) + c_3S(2\varphi_3 + 1) + S[2w(g) + S(u_1)] + c_4 \tag{28}$$

where c_4 is another arbitrary constant of integration.

Differentiating Eqs. (28), (25), and (22), respectively, one obtains

$$y_1'(\theta) = 2c_1\varphi_1(\theta) + 2c_2\varphi_2(\theta) + c_3[2\varphi_3(\theta) + 1] + 2w[g(\theta)] + S[u_1(\theta)] \tag{29a}$$

$$y_2'(\theta) = c_1\varphi_1'(\theta) + c_2\varphi_2'(\theta) + c_3\varphi_3'(\theta) + \frac{\mathrm{d}w[g(\theta)]}{\mathrm{d}\theta} \tag{29b}$$

$$y_3'(\theta) = -c_5\sin\theta + c_6\cos\theta + S[\cos\theta u_3(\theta)]\cos\theta + S[\sin\theta u_3(\theta)]\sin\theta \tag{29c}$$

Letting $\boldsymbol{z} = (y_1, y_1', y_2, y_2', y_3, y_3')^T$ and $\boldsymbol{c} = (c_1, c_2, c_3, c_4, c_5, c_6)^T$, the relevant equations are put in the form

$$\boldsymbol{z} = \Phi\boldsymbol{c} + r(\boldsymbol{u}) \tag{30}$$

where

$$\Phi = \begin{bmatrix} 2S(\varphi_1) & 2S(\varphi_2) & S(2\varphi_3 + 1) & 1 & 0 & 0 \\ 2\varphi_1 & 2\varphi_2 & 2\varphi_3 + 1 & 0 & 0 & 0 \\ \varphi_1 & \varphi_2 & \varphi_3 & 0 & 0 & 0 \\ \varphi_1' & \varphi_2' & \varphi_3' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos & \sin \\ 0 & 0 & 0 & 0 & -\sin & \cos \end{bmatrix} \tag{31}$$

and

$$\boldsymbol{r}(\boldsymbol{u}) = \begin{bmatrix} S[2w(g) + S(u_1)] \\ 2w(g) + S(u_1) \\ w(g) \\ \dfrac{\mathrm{d}w(g)}{\mathrm{d}\theta} \\ S(u_3\cos)\sin - S(u_3\sin)\cos \\ S(u_3\cos)\cos + S(u_3\sin)\sin \end{bmatrix} \tag{32}$$

Expression (31) is a fundamental matrix solution associated with Eq. (9). Its simplicity is appealing. The constants in Eq. (32) should be chosen so that $r[u(\theta_0)] = 0$. For this reason $\boldsymbol{z}(\theta_0) = \Phi(\theta_0)\boldsymbol{c}$, and one can write Eq. (30) as

$$\boldsymbol{z}(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1}\boldsymbol{z}(\theta_0) + \boldsymbol{r}[\boldsymbol{u}(\theta)] \tag{33}$$

One can use this fundamental matrix Φ in either Eq. (10) or Eq. (33). One must invert the matrix $\Phi(\theta)$ and evaluate this inverse at θ_0 to have the state transition matrix $\Phi(\theta)\Phi(\theta_0)^{-1}$.

VI. Formation of the State Transition Matrix

A. Some Identities

The normalized Wronskian (18) can be viewed as an identity in φ_1 and φ_2. Two other useful identities are the following:

$$2S(\varphi_1) + \varphi_1\varphi_3' - \varphi_3\varphi_1' = 0 \tag{34a}$$

$$2S(\varphi_2) + \varphi_2\varphi_3' - \varphi_3\varphi_2' = 0 \tag{34b}$$

These can be justified by substituting the expression for φ_3 in Eq. (26) and its derivative into the left-hand sides of Eqs. (34), simplifying, and applying the identity (18).

B. Adjoint System

Associated with the original system (9) is the adjoint system

$$\boldsymbol{\lambda}' = -A(\theta)^T \boldsymbol{\lambda} \tag{35}$$

It is useful to introduce the adjoint system and to find a fundamental matrix solution associated with it to apply the following known result.

Theorem: If $\Phi(\theta)$ is any fundamental matrix solution associated with the original system (9) and $\Psi(\theta)$ is any fundamental matrix solution associated with the adjoint system (35), then $\Psi(\theta)^T\Phi(\theta)$ is a constant matrix.

Proof:

$$[\Psi(\theta)^T\Phi(\theta)]' = \Psi'(\theta)^T\Phi(\theta) + \Psi(\theta)^T\Phi'(\theta)$$
$$= -[A(\theta)^T\Psi(\theta)]^T\Phi(\theta) + \Psi(\theta)^T A(\theta)\Phi(\theta) = 0 \qquad \square$$

In detail, the adjoint equations are

$$\lambda_1' = 0 \tag{36a}$$
$$\lambda_2' = -\lambda_1 + 2\lambda_4 \tag{36b}$$
$$\lambda_3' = -G[\omega(\theta)]\lambda_4 \tag{36c}$$
$$\lambda_4' = -2\lambda_2 - \lambda_3 \tag{36d}$$
$$\lambda_5' = \lambda_6 \tag{36e}$$
$$\lambda_6' = -\lambda_5 \tag{36f}$$

These equations can be combined to produce

$$\lambda_2'' - 2\lambda_4' = 0 \tag{37a}$$
$$\lambda_4'' - G[\omega(\theta)]\lambda_4 + 2\lambda_2' = 0 \tag{37b}$$
$$\lambda_6'' + \lambda_6 = 0 \tag{37c}$$

which are analogous to the homogeneous form of Eqs. (6). The analogous solutions are, therefore,

$$\lambda_2 = 2c_1S(\varphi_1) + 2c_2S(\varphi_2) + c_3S(2\varphi_3 + 1) + c_4 \tag{38a}$$
$$\lambda_4 = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 \tag{38b}$$
$$\lambda_6 = c_5\cos + c_6\sin \tag{38c}$$

and λ_1, λ_3, and λ_5 are found from Eqs. (36b), (36d), and (36f), respectively. A fundamental matrix solution is, therefore,

$$\Psi = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 2S(\varphi_1) & 2S(\varphi_2) & S(2\varphi_3+1) & 1 & 0 & 0 \\ -[4S(\varphi_1)+\varphi_1'] & -[4S(\varphi_2)+\varphi_2'] & -[2S(2\varphi_3+1)+\varphi_3'] & -2 & 0 & 0 \\ \varphi_1 & \varphi_2 & \varphi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin & -\cos \\ 0 & 0 & 0 & 0 & \cos & \sin \end{bmatrix} \tag{39}$$

C. State Transition Matrix

According to the preceding theorem $\Psi(\theta)^T\Phi(\theta) = C$ where C is a constant matrix. Performing this multiplication and simplifying through the use of the identities (18) and (34), one obtains

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{40}$$

This matrix is orthogonal, and so $C^{-1} = C^T$, consequently,

$$\Phi(\theta)^{-1} = C^T\Psi(\theta)^T \tag{41}$$

Performing this multiplication, one finds that

$$\Phi^{-1} = \begin{bmatrix} 0 & -2S(\varphi_2) & 4S(\varphi_2)+\varphi_2' & -\varphi_2 & 0 & 0 \\ 0 & 2S(\varphi_2) & -[4S(\varphi_1)+\varphi_1'] & \varphi_1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 1 & -S(2\varphi_3+1) & 2S(2\varphi_3+1)+\varphi_3' & -\varphi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos & -\sin \\ 0 & 0 & 0 & 0 & \sin & \cos \end{bmatrix} \tag{42}$$

This expression is readily checked by multiplying Eqs. (31) and (42). There is no further need of matrix inversion. The goal has been accomplished. The state transition matrix $\Phi(\theta)\Phi(\theta_0)^{-1}$ is completely determined.

VII. Linearized Rendezvous Near Keplerian Orbit in a Newtonian Gravitational Field

The work is now applied to the problem of linearized rendezvous near a Keplerian orbit in an inverse-square law gravitational field using the Tschauner–Hempel equations.

A. State Transition Matrix for the Tschauner–Hempel Equations

Because the goal is to find specific functions φ_1, φ_2, and φ_3 for the aforementioned state transition matrix, it is sufficient to assume unpowered flight, i.e., $u(\theta) = 0$ identically. For this reason the solutions (28), (25), and (22) describing the state variables become

$$y_1(\theta) = 2c_1S[\varphi_1(\theta)] + 2c_2S[\varphi_2(\theta)] + c_3S[2\varphi_3(\theta) + 1] + c_4 \tag{43a}$$
$$y_2(\theta) = c_1\varphi_1(\theta) + c_2\varphi_2(\theta) + c_3\varphi_3(\theta) \tag{43b}$$
$$y_3(\theta) = c_5\cos\theta + c_6\sin\theta \tag{43c}$$

The Tschauner–Hempel equations are a special case of Eqs. (6) where

$$G[\omega(\theta)] = 3/(1 + e\cos\theta) \tag{44}$$

and e denotes the eccentricity of the Keplerian orbit of the satellite.

Assuming $\boldsymbol{u}(\theta) = 0$ identically, integrating Eq. (6a), denoting the constant of integration as c_3, substituting for y_1', and inserting expression (44), the differential Eq. (6b) becomes

$$y_2'' + \frac{1 + 4e\cos\theta}{1 + e\cos\theta}y_2 = -2c_3 \tag{45}$$

This differential equation was solved for $e < 1$ by Tschauner and Hempel,[58] Tschauner,[60] Shulman and Scott,[63] and Weiss[66] in terms of the eccentric anomaly. Care should be taken to avoid singularities if one wants to have the solutions written in terms of θ. Lawden[61] and Carter and Humi[68] found that

$$\varphi_1(\theta) = \rho(\theta)\sin\theta \tag{46}$$

is a solution of the homogeneous form of Eq. (45), where the notation

$$\rho(\theta) = 1 + e\cos\theta \tag{47}$$

is used for brevity. By reduction of order a second solution of the form

$$\varphi_2(\theta) = \varphi_1(\theta)I(\theta) \tag{48}$$

can be found in terms of the expression

$$I(\theta) = S\{1/[\rho(\theta)\sin\theta]^2\} \tag{49}$$

This integral was used by Lawden[62,70] as early as 1954 and as recently as 1993. Unfortunately the integral $I(\theta)$ is singular at zero and integral multiples of π. Carter[71] removed the singularities by using an alternative solution:

$$\varphi_2(\theta) = 2e\rho(\theta)\sin(\theta)J(\theta) - \cos\theta/\rho(\theta) \tag{50}$$

that is written in terms of a new integral:

$$J(\theta) = S[\cos\theta/\rho(\theta)^3] \tag{51}$$

Defining φ_1 and φ_2 by Eqs. (46) and (50), respectively, it is straightforward to show that condition (18) is satisfied. If one selects the antiderivatives

$$S[\varphi_1(\theta)] = -\cos\theta - (e/2)\cos^2\theta \tag{52}$$

$$S[\varphi_2(\theta)] = -\rho(\theta)^2 J(\theta) \tag{53}$$

then from Eq. (26) one obtains

$$\varphi_3(\theta) = -2\rho(\theta)\sin\theta J(\theta) - \cos^2\theta/\rho(\theta) - \cos^2\theta \tag{54}$$

The general solution (43b) of the differential equation (45) is, therefore, given through Eqs. (46), (50), and (54). The solution (43c) for y_3 is unaffected by any of these expressions.

Now that the general solutions for y_2 and y_3 have been found, it remains to find the general solution for y_1. Straightforward antidifferentiation of Eq. (54) leads to the following crucial term of Eq. (43a),

$$S[2\varphi_3(\theta) + 1] = (2/e)[\rho(\theta)^2 J(\theta) - \sin\theta] - \sin\theta\cos\theta \tag{55}$$

This expression does not apply for circular orbits because of the troublesome appearance of e in the denominator. This annoyance is typical in the Tschauner–Hempel problem.

To avoid the singularity at $e = 0$, the new antiderivative $J(\theta)$ is replaced by a newer antiderivative:

$$K(\theta) = S[\sin^2\theta/\rho(\theta)^4] \tag{56}$$

through the substitution

$$J(\theta) = \sin\theta/\rho(\theta)^3 - 3eK(\theta) \tag{57}$$

In this manner Eq. (55) can be replaced by

$$S[2\varphi_3(\theta) + 1] = -6\rho(\theta)^2 K(\theta) - \sin\theta\cos\theta - 2\sin\theta\cos\theta/\rho(\theta) \tag{58}$$

which is continuous at $e = 0$. The preferred form of Eq. (43a) employs Eqs. (52), (53), (57), and (58). In this manner a complete solution of the Tschauner–Hempel equations has been found in terms of the expression $K(\theta)$ that is usable for circular, elliptical, parabolic, or hyperbolic orbits.

For convenience, the aforementioned results are summarized in terms of $K(\theta)$ as follows:

$$\varphi_1(\theta) = \rho(\theta)\sin\theta, \qquad \varphi_1'(\theta) = \cos\theta + e(\cos^2\theta - \sin^2\theta)$$

$$\varphi_2(\theta) = -6e^2\varphi_1(\theta)K(\theta) + 2e\sin^2\theta/\rho(\theta)^2 - \cos\theta/\rho(\theta)$$

$$\varphi_2'(\theta) = -6e^2\varphi_1'(\theta)K(\theta) + 2e\sin\theta(2\cos\theta - 3e\sin^2\theta + 2e)/\rho(\theta)^3 + \sin\theta/\rho(\theta)^2$$

$$\varphi_3(\theta) = 6e\varphi_1(\theta)K(\theta) - 2\sin^2\theta/\rho(\theta)^2 - \cos^2\theta/\rho(\theta) - \cos^2\theta$$

$$\varphi_3'(\theta) = 6e\varphi_1'(\theta)K(\theta) + 6e\sin^3\theta/\rho(\theta)^3 - 4\sin\theta(e + \cos\theta)/\rho(\theta)^3 + \sin\theta\cos\theta(2 + e\cos\theta)\rho(\theta)^2 + 2\sin\theta\cos\theta$$

$$S[\varphi_1(\theta)] = -\cos\theta[1 + (e/2)\cos\theta]$$

$$S[\varphi_2(\theta)] = 3e\rho(\theta)^2 K(\theta) - \sin\theta/\rho(\theta)$$

$$S[2\varphi_3(\theta) + 1] = -6\rho(\theta)^2 K(\theta) - 2\sin\theta\cos\theta/\rho(\theta) - \sin\theta\cos\theta \tag{59}$$

Substitution of these expressions into Eqs. (31) and (42) defines the Tschauner–Hempel state transition matrix. For unpowered flight, the transition from the state $z(\theta_0)$ to $z(\theta)$ is accomplished by

$$z(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1}z(\theta_0) \tag{60}$$

From the state vector $z(\theta)$ one finds $y(\theta)$ and $y'(\theta)$ in terms of $y(\theta_0)$ and $y'(\theta_0)$. The odd entries of $z(\theta)$ comprise the vector $y(\theta)$ and the even entries comprise the vector $y'(\theta)$. It is important to remember to transfer back to the actual relative position vector $x(\theta)$ via Eq. (12). The actual relative velocity is then

$$\frac{dx[\theta(t)]}{dt} = x'(\theta)\omega(\theta) \tag{61}$$

Performing the operations and simplifying, the actual relative position and velocity, respectively, are given as follows:

$$x(\theta) = \left(L^{\frac{3}{2}}/\mu\right)[y(\theta)/\rho(\theta)] \tag{62}$$

$$\frac{dx[\theta(t)]}{dt} = \frac{\mu}{L^{\frac{3}{2}}}[e\sin\theta\, y(\theta) + \rho(\theta)y'(\theta)] \tag{63}$$

B. Evaluation of $K(\theta)$

This form of solution of the Tschauner–Hempel equations and state transition matrix are dependent on the integral $K(\theta)$. The most practical and flexible approach to the evaluation of $K(\theta)$ in actual problems is probably direct numerical integration because the closed-form solutions vary with the type of orbit of the satellite.

For circular orbits, $e = 0$ and

$$K(\theta) = S(\sin^2\theta) = \tfrac{1}{2}(\theta - \sin\theta\cos\theta) \tag{64}$$

For parabolic orbits, $e = 1$ and

$$K(\theta) = S[\sin^2\theta/(1+\cos\theta)^4] = \tfrac{1}{4}S[\tan^2(\theta/2)] = \tfrac{1}{2}[\tan(\theta/2) - (\theta/2)] \tag{65}$$

This expression is simpler than the corresponding result for the integral $J(\theta)$ (Ref. 71).

If $0 < e < 1$, the orbit is elliptical and the closed-form evaluation of $K(\theta)$ is cumbersome. If one transforms the variable θ to the eccentric anomaly E by the relationship

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E} \tag{66}$$

where $\sin\theta$ and $\sin E$ have the same algebraic sign, the integral simplifies to

$$K(\theta) = (1 - e^2)^{-\frac{5}{2}} S[\sin^2 E(1 - e\cos E)] \tag{67}$$

where the integration is performed with respect to E instead of θ.

This integral is easily evaluated in closed form

$$K(\theta) = (1 - e^2)^{-\frac{5}{2}}\left[\tfrac{1}{2}E - \tfrac{1}{2}\sin E\cos E - (e/3)\sin^3 E\right] \tag{68}$$

Similarly, if $e > 1$, the orbit is hyperbolic and one can transform from the true anomaly θ to the hyperbolic anomaly H by the relationship

$$\cos\theta = \frac{e - \cosh H}{e\cosh H - 1} \tag{69}$$

where $\sin\theta$ and $\sin H$ have the same algebraic sign. This leads to the integral

$$K(\theta) = (e^2 - 1)^{-\frac{5}{2}} S[\sinh^2 H(e\cosh H - 1)] \tag{70}$$

which is also easily evaluated:

$$K(\theta) = (e^2 - 1)^{-\frac{5}{2}} \left(\tfrac{1}{2} H - \tfrac{1}{2} \sinh H \cosh H + (e/3) \sinh^3 H\right) \tag{71}$$

Any constant can be added to each of the expressions (64), (65), (68), and (71). None of these expressions is cumbersome.

VIII. Conclusions

Much work has been done by many researchers in the derivation and solution of various linearized rendezvous equations and in the formation of a subsequent state transition matrix. This paper presents the structure of solution of a generalized linear rendezvous problem, valid in any central force field including the Newtonian field, and uses this structure to establish a general state transition matrix without unnecessary complexity. These results were applied to the problem of rendezvous of a spacecraft with a satellite in an arbitrary Keplerian orbit using the Tschauner–Hempel equations. The solution of this problem and the related state transition matrix are concise, devoid of singularities, and more flexible than earlier forms found in the literature. Other applications are possible, the most obvious being the inclusion of oblateness effects of the planet on its gravitational field attracting the satellite.

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